CHAPTER 11

INTRODUCTION TO THE LAGRANGIAN AND HAMILTONIAN FORMULATIONS FOR CONTINUOUS SYSTEMS AND FIELDS

All the formulations of mechanics discussed up to this point have been devised for treating systems with a finite or at most a denumerably infinite number of degrees of freedom. There are some mechanical problems, however, which involve continuous systems, as, for example, the problem of a vibrating elastic solid. Here each point of the continuous solid partakes in the oscillations, and the complete motion can only be described by specifying the position coordinates of *all* points. It is not difficult to modify the previous formulations of mechanics so as to handle such problems. The most direct method is to approximate the continuous system by one containing discrete particles, and then examine the change in the equations describing the motion as the continuous limit is approached.

11–1 The transition from a discrete to a continuous system. We shall apply this procedure to an infinitely long elastic rod which can undergo small longitudinal vibrations, i.e., oscillatory displacements of the particles of the rod parallel to the axis of the rod. A system composed of discrete particles which approximates the continuous rod is an infinite chain of

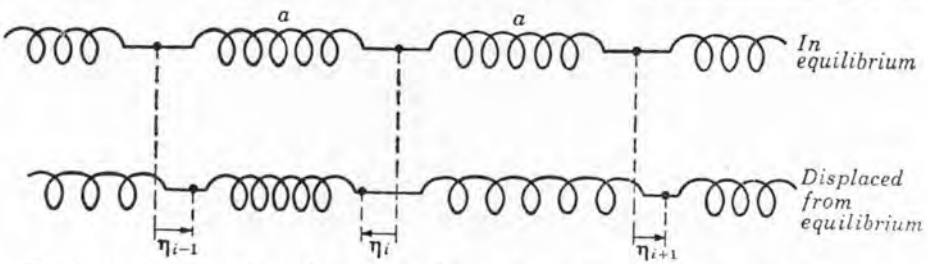

FIG. 11–1. A discrete system of equal mass points connected by springs, as an approximation to a continuous elastic rod.

equal mass points spaced a distance a apart and connected by uniform massless springs having force constants k (cf. Fig. 11–1). It will be assumed that the mass points can move only along the length of the chain. The discrete system will be recognized as an extension of the linear polyatomic molecule discussed in the preceding chapter. We can therefore obtain the equations describing the motion by the customary techniques

for small oscillations. Denoting the displacement of the ith particle from its equilibrium position by η_i, the kinetic energy is:

$$T = \frac{1}{2}\sum_i m\dot{\eta}_i^2, \tag{11-1}$$

where m is the mass of each particle. The corresponding potential energy is the sum of the potential energies of each spring as the result of being stretched or compressed from its equilibrium length (cf. Sec. 10–4):

$$V = \frac{1}{2}\sum_i k(\eta_{i+1} - \eta_i)^2. \tag{11-2}$$

That Eq. (11–2) is the correct potential energy may be seen also by calculating directly the force on the ith particle and comparing it with the force predicted from V. The force due to the spring on the right of the particle will be $k(\eta_{i+1}-\eta_i)$, while the spring on the left exerts the force $-k(\eta_i-\eta_{i-1})$, so that the total force is

$$F_i = k(\eta_{i+1} - \eta_i) - k(\eta_i - \eta_{i-1}),$$

which agrees with $F_i = -\dfrac{\partial V}{\partial \eta_i}$ as obtained from Eq. (11–2). Combining Eqs. (11–1) and (11–2), the Lagrangian for the system is

$$L = T - V = \frac{1}{2}\sum_i \left(m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right), \tag{11-3}$$

which can also be written as

$$L = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right] = \sum_i aL_i. \tag{11-4}$$

The resulting Lagrange equations of motion for the coordinates η_i are

$$\frac{m}{a}\ddot{\eta}_i - ka\left(\frac{\eta_{i+1} - \eta_i}{a^2}\right) + ka\left(\frac{\eta_i - \eta_{i-1}}{a^2}\right) = 0. \tag{11-5}$$

The particular form of L in Eq. (11–4), and of the corresponding equations of motion, has been chosen for convenience in going to the limit of a continuous rod as a approaches zero. It is clear that m/a reduces to μ, the mass per unit length of the continuous system, but the limiting value of ka may not be so obvious. For an elastic rod obeying Hooke's law it will be remembered that the extension of the rod *per unit length* is directly pro-

portional to the force or tension exerted on the rod, a relation that can be written as

$$F = Y\xi,$$

where ξ is the elongation per unit length and Y is Young's modulus. Now the extension of a length a of a discrete system, per unit length, will be $\xi = (\eta_{i+1} - \eta_i)/a$. The force necessary to stretch the spring by this amount is

$$F = k(\eta_{i+1} - \eta_i) = ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right),$$

so that ka must correspond to the Young's modulus of the continuous rod. In going from the discrete to the continuous case, the integer index i identifying the particular mass point becomes the continuous position coordinate x; instead of the variable η_i we have $\eta(x)$. Further, the quantity

$$\frac{\eta_{i+1} - \eta_i}{a} = \frac{\eta(x + a) - \eta(x)}{a}$$

occurring in L_i obviously approaches the limit

$$\frac{d\eta}{dx},$$

as a, playing the role of dx, approaches zero. Finally, the summation over a discrete number of particles becomes an integral over x, the length of the rod, and the Lagrangian (11–4) appears as

$$L = \frac{1}{2}\int\left(\mu\dot{\eta}^2 - Y\left(\frac{d\eta}{dx}\right)^2\right)dx. \tag{11–6}$$

In the limit as a goes to zero, the last two terms in the equation of motion (11–5) become

$$\lim_{a\to 0} -\frac{Y}{a}\left\{\left(\frac{d\eta}{dx}\right)_x - \left(\frac{d\eta}{dx}\right)_{x-a}\right\},$$

which clearly defines a second derivative of η. Hence the equation of motion for the continuous elastic rod is

$$\mu\frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0. \tag{11–7}$$

This simple example is sufficient to illustrate the salient features of the transition from a discrete to a continuous system. The most important fact to grasp is the role played by the position coordinate x. It is *not* a generalized coordinate; it serves merely as a continuous index replacing the

discrete i. Just as each value of i corresponded to a different one of the generalized coordinates, η_i, of the system, so here for each value of x there is a generalized coordinate $\eta(x)$. Since η depends also on the continuous variable t, we should perhaps write more accurately $\eta(x, t)$, indicating that x, like t, can be considered as a parameter entering into the Lagrangian. If the continuous system were three-dimensional, rather than one-dimensional as here, the generalized coordinates would be distinguished by three continuous indices x, y, z, and would be written as $\eta(x, y, z, t)$. Eq. (11–6) also shows that the Lagrangian appears as an integral over the continuous index x; in the corresponding three-dimensional case the Lagrangian would have the form

$$L = \iiint \mathfrak{L}\, dx\, dy\, dz, \tag{11–8}$$

where $\mathfrak{L}$ is known as the *Lagrangian density*. For the longitudinal vibrations of the continuous rod the Lagrangian density is

$$\mathfrak{L} = \frac{1}{2}\left\{\mu\left(\frac{\partial\eta}{\partial t}\right)^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right\}, \tag{11–9}$$

corresponding to the continuous limit of the quantity L_i appearing in Eq. (11–4). It is the Lagrangian density, rather than the Lagrangian itself, which will be used to describe the motion of the system.

11–2 The Lagrangian formulation for continuous systems. It will be noted from Eq. (11–9) that $\mathfrak{L}$ for the elastic rod, besides being a function of $\dot{\eta} \equiv \dfrac{\partial\eta}{\partial t}$, also involves a spatial derivative of η, namely, $\dfrac{\partial\eta}{\partial x}$; x and t thus play a similar role as parameters of the Lagrangian density. Of course, $\mathfrak{L}$ in specific cases may be a function of η itself, and may also involve t and x explicitly. In any general three-dimensional continuous system the Lagrangian density will appear as a function of the form

$$\mathfrak{L} = \mathfrak{L}\left(\eta, \frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y}, \frac{\partial\eta}{\partial z}, \frac{\partial\eta}{\partial t}, x, y, z, t\right). \tag{11–10}$$

The Lagrangian played an important role in the mechanics of discrete systems because we could obtain the equations of motion from it. For continuous systems, however, the equations of motion for $\eta(x, t)$ are given directly in terms of the Lagrangian density, $\mathfrak{L}$. Fundamentally, the equations of motion must come from Hamilton's principle, which now takes the form

$$\delta I = \delta \int_1^2 \iiint \mathfrak{L}\, dx\, dy\, dz\, dt = 0. \tag{11–11}$$

The nature of the variation involved in Eq. (11–11) differs only slightly from that previously discussed. There can be no variation of the parameters x, y, z; the virtual displacements constructing the varied paths are for constant xyz as well as for constant t. The variation process affects neither the limits of integration over time, nor the region of volume integration. Just as the variation of η is taken to be zero at the end points t_1 and t_2, so the variation of η on the surface of the volume of integration is also to be zero. The technique of converting the variational principle into an ordinary extremum problem by labelling the varied paths with some parameter α works as well here as in the discrete case. Sufficient experience has been gained by now in handling the δ-variation so that we can dispense with the parameter notation and deal directly with the variations themselves, remembering always that

$$\delta \rightarrow d\alpha \frac{\partial}{\partial \alpha}.$$

Since the Lagrangian density is a function not only of η and $\dot{\eta}$ but also of the spatial derivatives of η, the variation of $\mathfrak{L}$ can be written

$$\delta \mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \eta} \delta\eta + \frac{\partial \mathfrak{L}}{\partial \dot{\eta}} \delta\dot{\eta} + \sum_{k=1}^{3} \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x_k}\right)} \delta\left(\frac{\partial \eta}{\partial x_k}\right), \tag{11–12}$$

where for convenience the spatial coordinates have been changed from xyz to $x_1x_2x_3$. Hamilton's principle therefore becomes

$$\int_1^2 \iiint \left[\frac{\partial \mathfrak{L}}{\partial \eta} \delta\eta + \frac{\partial \mathfrak{L}}{\partial \dot{\eta}} \delta\dot{\eta} + \sum_{k=1}^{3} \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x_k}\right)} \delta\left(\frac{\partial \eta}{\partial x_k}\right)\right] dx_1\, dx_2\, dx_3\, dt = 0. \tag{11–13}$$

Making use of the same integration by parts occurring in the derivation of the ordinary Lagrange equations, we obtain the relation

$$\int_1^2 \frac{\partial \mathfrak{L}}{\partial \dot{\eta}} \delta\dot{\eta}\, dt = -\int_1^2 \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\eta}}\right) \delta\eta\, dt.$$

The integrals involving the spatial derivatives of η can be manipulated in a similar manner. Interchanging the derivative with respect to x_k and the δ-variation, we have

$$\int \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x_k}\right)} \delta\left(\frac{\partial \eta}{\partial x_k}\right) dx_k = \int \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x_k}\right)} \frac{\partial\, \delta\eta}{\partial x_k} dx_k. \tag{11–14}$$

An integration by parts now converts the integral into *

$$\frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \eta}{\partial x_k}\right)}\delta\eta - \int \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \eta}{\partial x_k}\right)}\right)\delta\eta \, dx_k. \tag{11-15}$$

The integrated term in Eq. (11–15) vanishes, as in the time integral, because the variation process is such that $\delta\eta$ vanishes at the extremities of the spatial integration. There is some difficulty, in principle at least, if the system is infinite in extent. Very often in such systems the disturbance measured by η falls off sufficiently fast so that the integrated term is zero at infinity regardless of the nature of the variation. In any case, the integral can always be taken formally as over a finite region and, after dropping the integrated term, the volume can be allowed to become infinite. Consequently, Hamilton's principle takes on the form:

$$\int_1^2 \iiint \delta\eta \left[\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \sum_{k=1}^{3} \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \eta}{\partial x_k}\right)}\right)\right] dx_1\, dx_2\, dx_3\, dt = 0. \tag{11-16}$$

The integral can vanish identically only if the separate coefficients of the independent variations $\delta\eta(x_1, x_2, x_3, t)$ vanish, yielding the equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} + \sum_{k=1}^{3} \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial \eta}{\partial x_k}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0. \tag{11-17}$$

A system of n discrete degrees of freedom will have n Lagrange equations of motion; for the continuous system with an infinite number of degrees of freedom we seem to obtain only one Lagrange equation! It must be remembered, however, that the equation of motion for η_i is a differential equation involving the time only, and in that sense Eq. (11–17) furnishes a separate equation of motion for each value of x_k. The continuous nature of the indices x_k appears in that Eq. (11–17) is a partial differential equation in the four variables $x_1x_2x_3$, and t, yielding η as $\eta(x_1, x_2, x_3, t)$.

* The change from a partial derivative with respect to x_k in (11–14) to a total derivative in (11–15) may occasion some difficulty. In the first case a partial derivative is used to indicate that η is a function not only of x_k but of t and the other spatial coordinates. Use of a partial derivative in (11–15) might have implied that only the explicit dependence of $\mathcal{L}$ on x_k was in question. To emphasize that the derivative must also involve the implicit dependence on x_k through η, we have therefore written a total derivative in (11–15). In any case, the operations actually to be performed are quite clear and unequivocal.

The form of the Lagrangian density and the subsequent equations of motion have been discussed assuming that each point in the system can suffer only one type of displacement, indicated by η. In a more complicated problem, such as the vibration of an elastic solid, a particle will, of course, undergo displacements along all three axes x_1, x_2, and x_3. In such case there will be three types of generalized coordinates, which we may denote by an integer index j: $\eta_j(x_1, x_2, x_3, t)$. In a general problem we must therefore expect to have generalized coordinates characterized by both a discrete index (or set of indices) and the continuous space indices $x_1x_2x_3$. The Lagrangian density $\mathfrak{L}$ will be a function of all the generalized coordinates and their space and time derivatives. For each $\eta_j(x_1, x_2, x_3, t)$ there will be an equation of motion of the form:

$$\frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\eta}_j} + \sum_k \frac{d}{dx_k}\left(\frac{\partial \mathfrak{L}}{\partial\left(\dfrac{\partial \eta_j}{\partial x_k}\right)}\right) - \frac{\partial \mathfrak{L}}{\partial \eta_j} = 0, \quad j = 1, 2, \ldots \tag{11-18}$$

Considerable simplification in notation is obtained by introducing a quantity known as a *functional derivative** or *variational derivative*. The functional derivative of the Lagrangian L with respect to η_j is defined as

$$\frac{\delta L}{\delta \eta_j} = \frac{\partial \mathfrak{L}}{\partial \eta_j} - \sum_{k=1}^{3} \frac{d}{dx_k}\frac{\partial \mathfrak{L}}{\partial\left(\dfrac{\partial \eta_j}{\partial x_k}\right)}. \tag{11-19}$$

A similar definition holds for the functional derivative of L with respect to $\dot{\eta}_j$, but since $\mathfrak{L}$ does not depend on the gradients of $\dot{\eta}_j$, we have simply

$$\frac{\delta L}{\delta \dot{\eta}_j} = \frac{\partial \mathfrak{L}}{\partial \dot{\eta}_j}. \tag{11-20}$$

The great advantage of the functional derivative notation is that it enables us in effect to forget the complicating dependence of $\mathfrak{L}$ on the spatial derivatives of η. Thus the δ-variation of L can be seen from Eqs. (11-8), (11-12), and (11-15) to be equal to

$$\delta L = \int \sum_j \left(\frac{\partial \mathfrak{L}}{\partial \eta_j}\delta\eta_j - \sum_k \frac{d}{dx_k}\frac{\partial \mathfrak{L}}{\partial\left(\dfrac{\partial \eta_j}{\partial x_k}\right)}\delta\eta_j + \frac{\partial \mathfrak{L}}{\partial \dot{\eta}_j}\delta\dot{\eta}_j\right) dV \tag{11-21}$$

(where dV is an element of volume). In functional derivative notation, this result becomes simply

* The name is applied because the functional derivative of, say, L with respect to η gives the change in L due to a change in the value of the *function* $\eta(x)$ at a specific space point x, there being no change in the dependence of η on t.

$$\delta L = \int \sum_j \left(\frac{\delta L}{\delta \eta_j} \delta \eta_j + \frac{\delta L}{\delta \dot{\eta}_j} \delta \dot{\eta}_j \right) dV, \tag{11-22}$$

which is exactly what would have been obtained if the δ-variation were applied directly to Eq. (11–8), ignoring the dependence on the gradients of η. The equations of motion (11–18) similarly take on a simple form in terms of functional derivatives:

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\eta}_j} - \frac{\delta L}{\delta \eta_j} = 0, \tag{11-23}$$

greatly resembling the ordinary Lagrange equations in appearance.

While the functional derivative notation greatly simplifies some of the manipulation of the variational principles, its use tends to obscure the fact that the equations of motion are *partial differential* equations in space and time. It also singles out the time coordinate as different in nature from the spatial coordinates, whereas the derivation actually treats x_k and t in equal fashion as parameters appearing in $\mathfrak{L}$. This equal status reminds us of special relativity; in fact, the entire procedure slips very easily into a Lorentz covariant formulation. The product $dx_1\, dx_2\, dx_3\, dt$ is essentially an element of volume in world space and hence is an invariant under a Lorentz transformation. Hamilton's principle (11–11) is therefore automatically Lorentz invariant providing only that $\mathfrak{L}$ is a world scalar. The equation of motion, (11–17), in covariant notation becomes simply

$$\sum_{\mu=1}^{4} \frac{d}{dx_\mu} \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x_\mu} \right)} - \frac{\partial \mathfrak{L}}{\partial \eta} = 0, \tag{11-24}$$

which will be relativistically invariant providing $\mathfrak{L}$ and η are world scalars. Similarly, the covariance of the equations of motion (11–18) is assured if $\mathfrak{L}$ is a world scalar and the η_i's have some definite Lorentz transformation properties, e.g., are the components of a four-vector.

As a simple example of the Lagrangian procedure for obtaining the equation of motion, let us return to the longitudinal vibrations of a long elastic rod, treated in the previous section. The Lagrangian density is given by Eq. (11–9) and the various derivatives appearing in Eq. (11–17) are

$$\frac{\partial \mathfrak{L}}{\partial \eta} = 0, \qquad \frac{\partial \mathfrak{L}}{\partial \dot{\eta}} = \mu \dot{\eta}, \qquad \frac{\partial \mathfrak{L}}{\partial \left(\frac{\partial \eta}{\partial x} \right)} = -Y \frac{\partial \eta}{\partial x}.$$

Hence the equation of motion obtained from the Lagrangian density is:

$$\mu \frac{d^2\eta}{dt^2} - Y \frac{d^2\eta}{dx^2} = 0,$$

which agrees with (11–7), found earlier. Eq. (11–7) will be recognized as a one-dimensional wave equation with the propagation velocity:

$$v = \sqrt{\frac{Y}{\mu}}, \tag{11–25}$$

which is the well-known formula for the velocity of longitudinal elastic waves.

11–3 Sound vibrations in gases as an example of the Lagrangian formulation. To illustrate the Lagrangian procedure for handling the motion of continuous mechanical systems, we shall seek the equations of motion for the longitudinal vibrations of a gas. These vibrations, of course, constitute a *sound* field, and what we will obtain will be the wave equation for the propagation of sound. The displacement of the gas will be denoted by the vector $\boldsymbol{\eta}$, with components η_i, $i = 1, 2, 3$. Each point xyz in space will thus have three generalized coordinates associated with it. It will be assumed that the disturbance is always small, so that the pressure P and density μ differ only slightly from their equilibrium values P_0 and μ_0 respectively.

In a discrete system the problem is set up in the Lagrangian formulation by finding the kinetic and potential energies and writing the Lagrangian as the difference of these quantities. Here the Lagrangian we seek is the volume integral of a density $\mathfrak{L}$. The kinetic and potential energies can similarly be obtained as volume integrals of densities $\mathfrak{I}$ and $\mathfrak{V}$ respectively, with the relation

$$\mathfrak{L} = \mathfrak{I} - \mathfrak{V}. \tag{11–26}$$

The kinetic energy density presents no problem; bearing in mind that the displacements from equilibrium are small, we have

$$\mathfrak{I} = \frac{\mu_0}{2} \dot{\boldsymbol{\eta}}^2 = \frac{\mu_0}{2} (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2).$$

To obtain the potential energy density is a more difficult task. The potential energy of the gas is a measure of the work the gas can do in expanding against the pressure. Essentially, it arises from what the seventeenth century scientists were fond of calling the "spring" of the gas.

Consider a mass of gas M with equilibrium volume

$$V_0 = \frac{M}{\mu_0} \tag{11-27}$$

sufficiently small that $\mathcal{U}$ is constant over the volume. Then $\mathcal{U}V_0$ represents the potential energy of the quantity of gas. As a result of the sound disturbance, the volume changes from V_0 to $V_0 + \Delta V$. Now, in a change in volume dV the work performed on the system, i.e., the increase in the potential energy, is $-P\,dV$.* Hence the potential energy corresponding to a volume change from V_0 to $V_0 + \Delta V$ is

$$\mathcal{U}V_0 = -\int_{V_0}^{V_0+\Delta V} P\,dV.$$

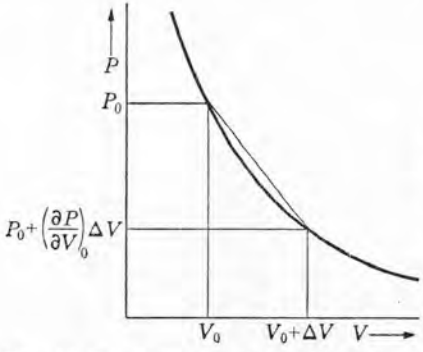

FIG. 11-2. Pressure-volume diagram for a gas.

It might be thought that since ΔV is small, the integral can be approximated by $P_0\,\Delta V$. As we shall see, this term actually does not contribute to the equations of motion. It is therefore necessary to go to the next approximation, in which the curve of P vs. V is replaced by a straight line in the region from V_0 to $V_0 + \Delta V$ (cf. Fig. 11-2):

$$\int_{V_0}^{V_0+\Delta V} P\,dV = P_0\,\Delta V + \frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_0 (\Delta V)^2. \tag{11-28}$$

To evaluate the derivative of P with respect to V, we must digress for a moment into thermodynamics. The first inclination might be to use Boyle's law,

$$PV = C, \tag{11-29}$$

for the relation between the pressure and the volume, and this was the procedure followed by Newton. It leads to the wrong result, however, because (11-29) assumes that the changes in pressure and volume occur

* The customary elementary derivation is as follows. The force exerted on an element of surface dA by the external system is $P\,dA$, pointing inwards. In expanding, the surface moves a distance dx outward along the normal, and the external work done is $-P\,dA\,dx = -P\,dV$.

isothermally. Actually, the vibrations of sound are almost always so rapid that there is no time for conduction to remove the heat developed and equalize the temperatures. The contractions and expansions instead take place *adiabatically*, i.e., without loss of heat. Under these conditions the relation between P and V is

$$PV^\gamma = C, \tag{11-30}$$

where γ is the constant ratio of the specific heats at constant pressure and volume.* Hence the desired derivative is

$$\left(\frac{\partial P}{\partial V}\right)_0 = -\frac{\gamma P_0}{V_0}. \tag{11-31}$$

It is convenient to express the change in volume in terms of the associated density change. Since $V = M/\mu$, the change in V is given by

$$\Delta V = -\frac{M}{\mu_0^2}\Delta\mu = -V_0\sigma, \tag{11-32}$$

where the fractional change in the density has been denoted by σ:

$$\mu = \mu_0(1 + \sigma). \tag{11-33}$$

Combining Eqs. (11-27, 28, 31, and 32) the potential energy density appears as

$$\mathcal{U} = P_0\sigma + \frac{\gamma P_0}{2}\sigma^2. \tag{11-34}$$

This is still not in the form useful for the Lagrangian; we have yet to express σ in terms of $\boldsymbol{\eta}$. Consider any finite volume V in space. The mass flowing out of this volume due to the small disturbance from equilibrium is given by

$$\mu_0 \int \boldsymbol{\eta} \cdot d\mathbf{A},$$

evaluated over the surface of the volume. The volume integral of the change in density must be exactly equal to this mass transport:

$$-\mu_0 \int \sigma\, dV = \mu_0 \int \boldsymbol{\eta} \cdot d\mathbf{A}. \tag{11-35}$$

By the divergence theorem the relation (11-35) can be written

$$-\int \sigma\, dV = \int \nabla \cdot \boldsymbol{\eta}\, dV,$$

* For derivation see M. W. Zemansky, *Heat and Thermodynamics*, McGraw-Hill, Chapter VI.

and since the equality holds for any arbitrary volume, we must have *

$$\sigma = -\nabla \cdot \boldsymbol{\eta}. \tag{11-36}$$

With this connection, the final form of the potential energy density is

$$\mathcal{U} = -P_0 \nabla \cdot \boldsymbol{\eta} + \frac{\gamma P_0}{2} (\nabla \cdot \boldsymbol{\eta})^2. \tag{11-37}$$

It can now be seen that the term in $\mathcal{U}$ linear in σ cannot contribute to the total potential energy. By Eq. (11–35) the volume integral of σ is minus the surface integral of $\boldsymbol{\eta}$, and if the surface completely encloses the system this must be zero, i.e., there is no transport of mass out of the system. That this term has a vanishing contribution to L is not yet sufficient reason to omit it from $\mathcal{U}$. Conceivably, the functional behavior of the term might still have an effect on the equation of motion. (It will be remembered that the covariant Hamiltonian of a system may be zero, but the equations of motion, of course, do not vanish.) We shall therefore retain the term for a few more steps.

The complete Lagrangian density can now be written as

$$\mathfrak{L} = \frac{1}{2} (\mu_0 \dot{\boldsymbol{\eta}}^2 + 2P_0 \nabla \cdot \boldsymbol{\eta} - \gamma P_0 (\nabla \cdot \boldsymbol{\eta})^2). \tag{11-38}$$

In obtaining the equations of motion, we will need the following derivatives:

$$\begin{aligned} \frac{\partial \mathfrak{L}}{\partial \dot{\eta}_i} &= \mu_0 \dot{\eta}_i, \\ \frac{\partial (\nabla \cdot \boldsymbol{\eta})}{\partial \left(\dfrac{\partial \eta_i}{\partial x_k} \right)} &= \delta_{ik}, \end{aligned} \tag{11-39}$$

and

$$\frac{\partial (\nabla \cdot \boldsymbol{\eta})^2}{\partial \left(\dfrac{\partial \eta_i}{\partial x_k} \right)} = 2(\nabla \cdot \boldsymbol{\eta})\, \delta_{ik}.$$

It follows from Eq. (11–39) that the term $2P_0 \nabla \cdot \boldsymbol{\eta}$ cannot contribute to the equations of motion and can finally be dropped from $\mathfrak{L}$:

$$\mathfrak{L} = \frac{1}{2} (\mu_0 \dot{\boldsymbol{\eta}}^2 - \gamma P_0 (\nabla \cdot \boldsymbol{\eta})^2). \tag{11-40}$$

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* Eq. (11–36) may be recognized from its more familiar form,

$$\dot{\mu} = -\nabla \cdot \mu \dot{\boldsymbol{\eta}},$$

as the *equation of continuity* for the gas flow.

The resultant equations of motion are now

$$\mu_0 \frac{\partial^2 \eta_i}{\partial t^2} - \gamma P_0 \frac{\partial(\nabla \cdot \boldsymbol{\eta})}{\partial x_i} = 0, \quad i = 1, 2, 3. \tag{11–41}$$

Eqs. (11–41) may be combined into one vector equation as

$$\mu_0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \gamma P_0 \nabla \nabla \cdot \boldsymbol{\eta} = 0. \tag{11–42}$$

In this guise the physical significance of the equation of motion may still not be very apparent, but we can easily reduce it to a familiar form. Take the divergence of Eq. (11–42), remembering that $\nabla \cdot \nabla$ is the Laplacian ∇^2, and $\nabla \cdot \boldsymbol{\eta} = -\sigma$:

$$\nabla^2 \sigma - \frac{\mu_0}{\gamma P_0} \frac{\partial^2 \sigma}{\partial t^2} = 0. \tag{11–43}$$

Eq. (11–43) is now readily recognized as the three-dimensional wave equation, with the velocity

$$v = \sqrt{\frac{\gamma P_0}{\mu_0}}, \tag{11–44}$$

which is the customary expression for the velocity of sound in gases. The Lagrangian density (11–40) thus correctly represents the propagation of sound waves in gases, and we have achieved our goal of describing the sound vibrations by a Lagrangian formulation.

11–4 The Hamiltonian formulation for continuous systems. It is possible to obtain a Hamiltonian formulation for systems with a continuous set of coordinates much as was done in Chapter 7 for discrete systems. To indicate the method of approach, we return briefly to the linear chain of mass points discussed in Section 11–1. Conjugate to each η_i there is a canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{\eta}_i} = a \frac{\partial L_i}{\partial \dot{\eta}_i}. \tag{11–45}$$

The Hamiltonian for the system is therefore

$$H \equiv \sum_i p_i \dot{\eta}_i - L = \sum_i a \frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - L,$$

or

$$H = \sum_i a \left(\frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - L_i \right). \tag{11–46}$$

It will be remembered that in the limit of the continuous rod, when a goes to zero, $L_i \to \mathfrak{L}$ and the summation in Eq. (11–46) becomes an integral:

$$H = \int dx \left(\frac{\partial \mathfrak{L}}{\partial \dot{\eta}} \dot{\eta} - \mathfrak{L} \right). \tag{11–47}$$

The individual canonical momenta p_i, as given by Eq. (11–45), vanish in the continuous limit, but we can define a *momentum density*, π, which remains finite:

$$\pi = \frac{\partial \mathfrak{L}}{\partial \dot{\eta}}.$$

Eq. (11–47) is in the form of a space integral over a *Hamiltonian density*, $\mathfrak{IC}$, defined by

$$\mathfrak{IC} = \pi \dot{\eta} - \mathfrak{L}. \tag{11–48}$$

It is clear, therefore, that in any general three-dimensional system with more than one type of generalized coordinate the total Hamiltonian will be a volume integral of a Hamiltonian density:

$$H = \iiint \mathfrak{IC}\, dx_1\, dx_2\, dx_3 = \iiint \left(\sum_k \pi_k \dot{\eta}_k - \mathfrak{L} \right) dx_1\, dx_2\, dx_3, \tag{11–49}$$

where

$$\pi_i = \frac{\partial \mathfrak{L}}{\partial \dot{\eta}_i} = \frac{\delta L}{\delta \dot{\eta}_i}. \tag{11–50}$$

Canonical field equations of motion can be obtained from the Hamiltonian density by following the same procedure used in Section 7–1. We consider $\mathfrak{IC}$ to be given as a function of the coordinates $\eta_k(x_j, t)$, the canonical momentum densities $\pi_k(x_j, t)$, the spatial derivatives of the coordinates $\frac{\partial \eta_k}{\partial x_j}$, and possibly the time t. An infinitesimal change in H will then be written as

$$dH =$$

$$\iiint \left\{ \sum_k \left[\frac{\partial \mathfrak{IC}}{\partial \eta_k} d\eta_k + \frac{\partial \mathfrak{IC}}{\partial \pi_k} d\pi_k + \sum_j \frac{\partial \mathfrak{IC}}{\partial \left(\frac{\partial \eta_k}{\partial x_j} \right)} d\left(\frac{\partial \eta_k}{\partial x_j} \right) \right] + \frac{\partial \mathfrak{IC}}{\partial t} dt \right\} dx_1\, dx_2\, dx_3.$$

The integral appearing here roughly resembles the volume integral occurring in Hamilton's principle, Eq. (11–13). As was done in that instance, we perform a parts integration on the integral

$$\int \frac{\partial \mathfrak{IC}}{\partial \left(\frac{\partial \eta_k}{\partial x_j} \right)} d\left(\frac{\partial \eta_k}{\partial x_j} \right) dx_j.$$

The integrated term can be made to vanish by making the volume of integration larger than the system. Alternatively, the region of integration can be made so large that the values of η and $\mathfrak{H}$ on the surface at infinity are small enough to make the integrated term negligible. As a result dH can be written

$$dH =$$

$$\iiint \left\{ \sum_k \left[\frac{\partial \mathfrak{H}}{\partial \eta_k} d\eta_k + \frac{\partial \mathfrak{H}}{\partial \pi_k} d\pi_k - \sum_j \frac{d}{dx_j} \left(\frac{\partial \mathfrak{H}}{\partial \left(\frac{\partial \eta_k}{\partial x_j} \right)} \right) d\eta_k \right] + \frac{\partial \mathfrak{H}}{\partial t} dt \right\} dx_1\, dx_2\, dx_3. \tag{11-51}$$

By making use of the functional derivative as defined in Eq. (11–19) this expression can be reduced to

$$dH = \iiint \left\{ \sum_k \left(\frac{\delta H}{\delta \eta_k} d\eta_k + \frac{\delta H}{\delta \pi_k} d\pi_k \right) + \frac{\partial \mathfrak{H}}{\partial t} dt \right\} dx_1\, dx_2\, dx_3, \tag{11-52}$$

since $\mathfrak{H}$ is not a function of the spatial derivatives of π_k. Whenever we have to take a differential of a quantity whose density depends upon the spatial derivatives of η or π, a similar parts integration can be performed, and the various derivatives will be grouped so as to form functional derivatives. It is therefore always feasible to express the differentials from the start in terms of functional derivatives, which greatly streamlines the operations.

The differential of H can be expressed in another way, using its definition, Eq. (11–49), in terms of L:

$$dH = \iiint \left\{ \sum_k \left(\pi_k\, d\dot{\eta}_k + \dot{\eta}_k\, d\pi_k - \frac{\delta L}{\delta \eta_k} d\eta_k - \frac{\delta L}{\delta \dot{\eta}_k} d\dot{\eta}_k \right) - \frac{\partial \mathfrak{L}}{\partial t} dt \right\} dx_1\, dx_2\, dx_3. \tag{11-53}$$

From the definition of π_k, Eq. (11–50), the first and the last terms in the parentheses cancel, while in the third term

$$\frac{\delta L}{\delta \eta_k} = \frac{d}{dt} \frac{\delta L}{\delta \dot{\eta}_k} = \dot{\pi}_k,$$

from the Lagrange equations of motion (11–23). Hence Eq. (11–53) can be written

$$dH = \iiint \left\{ \sum_k \left(-\dot{\pi}_k\, d\eta_k + \dot{\eta}_k\, d\pi_k \right) - \frac{\partial \mathfrak{L}}{\partial t} dt \right\} dx_1\, dx_2\, dx_3. \tag{11-54}$$

Comparison of Eq. (11–54) with Eq. (11–52) results in the set of equations

$$\frac{\delta H}{\delta \eta_k} = -\dot{\pi}_k, \qquad \frac{\delta H}{\delta \pi_k} = \dot{\eta}_k, \tag{11–55}$$

and the identity

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathfrak{L}}{\partial t}.$$

Equations (11–55) are the analog of Hamilton's canonical equations for the continuous system; in terms of the Hamiltonian density they can be written

$$\frac{\partial \mathcal{H}}{\partial \eta_k} - \sum_j \frac{d}{dx_j}\left(\frac{\partial \mathcal{H}}{\partial\left(\frac{\partial \eta_k}{\partial x_j}\right)}\right) = -\dot{\pi}_k, \qquad \frac{\partial \mathcal{H}}{\partial \pi_k} = \dot{\eta}_k. \tag{11–56}$$

In this form they lose much of their symmetry, since $\mathcal{H}$ is not a function of the gradients of π_k.

As a simple example of the Hamiltonian procedure, we can examine the sound vibrations in a gas, as discussed in the previous section. The momentum density is simply

$$\pi_k = \mu_0 \dot{\eta}_k,$$

or, forming a vector momentum density:

$$\boldsymbol{\pi} = \mu_0 \dot{\boldsymbol{\eta}},$$

and the Hamiltonian density is (cf. Eq. (11–40)):

$$\mathcal{H} = \boldsymbol{\pi} \cdot \dot{\boldsymbol{\eta}} - \mathfrak{L} = \frac{\boldsymbol{\pi}^2}{2\mu_0} + \frac{P_0 \gamma}{2} (\nabla \cdot \boldsymbol{\eta})^2. \tag{11–57}$$

The Hamiltonian density is thus equal to the sum of the kinetic and potential energy densities:

$$\mathcal{H} = \mathfrak{I} + \mathcal{U},$$

and can therefore be identified in this case as an energy density.* The canonical equations obtained from $\mathcal{H}$ are

$$\dot{\eta}_k = \frac{\pi_k}{\mu_0}$$

(which merely repeats the definition of π_k), and

$$-\dot{\pi}_k = -\frac{d}{dx_k}(P_0 \gamma \nabla \cdot \boldsymbol{\eta}),$$

* The term linear in $\nabla \cdot \boldsymbol{\eta}$ has been omitted, for, as we have seen, it does not contribute to the total energy.

and the combination of these two sets of equations is identical with the equations of motion given by Eq. (11–41).

Much of the formal development of the Hamiltonian formulation — conservation theorems, Poisson brackets, etc. — can easily be recast in a form suitable for continuous systems. Thus, the modified Hamilton's principle becomes

$$\delta \int_1^2 \iiint \left\{ \sum_k \pi_k \dot{\eta}_k - \mathcal{H} \right\} dx_1\, dx_2\, dx_3\, dt = 0. \tag{11–58}$$

As an example of the conservation theorems, let us investigate the conditions under which the Hamiltonian is a constant of the motion. The total time derivative of H is

$$\frac{dH}{dt} = \iiint \left\{ \sum_k \left(\frac{\delta H}{\delta \eta_k} \dot{\eta}_k + \frac{\delta H}{\delta \pi_k} \dot{\pi}_k \right) + \frac{\partial \mathcal{H}}{\partial t} \right\} dx_1\, dx_2\, dx_3.$$

But by the equations of motion (11–55), this reduces to

$$\frac{dH}{dt} = \iiint \left\{ \sum_k \left(\frac{\delta H}{\delta \eta_k} \frac{\delta H}{\delta \pi_k} - \frac{\delta H}{\delta \pi_k} \frac{\delta H}{\delta \eta_k} \right) + \frac{\partial \mathcal{H}}{\partial t} \right\} dx_1\, dx_2\, dx_3,$$

so that

$$\frac{dH}{dt} = \iiint \frac{\partial \mathcal{H}}{\partial t}\, dx_1\, dx_2\, dx_3.$$

The total Hamiltonian is thus conserved if $\mathcal{H}$ (or H) is not an explicit function of time.

Similar techniques can be used to obtain an expression for the time derivative of any function G, not depending explicitly on time, which can be represented as the volume integral of a density function $\mathcal{G}$:

$$G = \iiint \mathcal{G}\, dx_1\, dx_2\, dx_3.$$

The total time derivative of G under these conditions is

$$\frac{dG}{dt} = \iiint \sum_k \left(\frac{\delta G}{\delta \eta_k} \dot{\eta}_k + \frac{\delta G}{\delta \pi_k} \dot{\pi}_k \right) dx_1\, dx_2\, dx_3.$$

By the equations of motion, this can be written

$$\frac{dG}{dt} = \iiint \sum_k \left(\frac{\delta G}{\delta \eta_k} \frac{\delta H}{\delta \pi_k} - \frac{\delta G}{\delta \pi_k} \frac{\delta H}{\delta \eta_k} \right) dx_1\, dx_2\, dx_3. \tag{11–59}$$

Now, the integral on the right in Eq. (11–59) is the exact analog of the Poisson bracket of G with H (cf. Eq. (8–42)), the summation over the generalized coordinates appearing as an integral over the continuous indices x_1, x_2, and x_3, and a discrete summation over the index k. We can therefore write

$$\frac{dG}{dt} = [G, H], \tag{11–60}$$

and it is clear that if G depends on time explicitly we obtain

$$\frac{dG}{dt} = [G, H] + \frac{\partial G}{\partial t},$$

in complete agreement with Eq. (8–58). Any integral function that is not an explicit function of time, and whose Poisson bracket with H vanishes, will thus be a constant of the motion. Note that the derivation is valid even when $\mathcal{G}$ is a function of the spatial derivatives of η or π, for this is exactly the situation the functional derivatives were designed to include.

The conservation theorems for integral quantities thus follow as in the ordinary theory. We have the same connection between the constants of the motion and the symmetry properties of the system as was found previously. It must be mentioned, however, that besides these "macroscopic" constants of the motion, there are "microscopic" conservation theorems. These deal directly with the densities rather than with the integrated quantities. One can obtain, for example, theorems which are essentially equations of continuity for the internal flow of energy and both linear and angular momentum. Unfortunately, to discuss such matters would lead us too far afield, and we shall have to refer the interested reader to the works cited at the end of the chapter.

11–5 Description of fields by variational principles. The Lagrangian and Hamiltonian formulations for a continuous set of generalized coordinates were developed in order to treat continuous mechanical systems, such as an elastic solid. But the formulations may also be used, even in the absence of a mechanical system, to describe the equations governing a *field*. Mathematically, a field is no more than one or more independent functions of space and time, and the generalized coordinates $\eta_j(x_1, x_2, x_3, t)$ fit this definition accurately. Indeed, fields occurring in physics often arose historically as the vibrations of some continuous system. The sound "field" in gases, after all, actually refers to the longitudinal vibrations of the gas particles, discussed in Section 11–3. Similarly, the electromagnetic field was long thought of in terms of the elastic vibrations of a

mysterious ether. Only in recent times was it realized (to quote an apt phrase of Professor S. L. Quimby) that the ether had no other role than being the subject of the verb "to undulate." We can equally well recognize that the variational principle procedures developed in the previous sections also stand independent of the notion of a continuous mechanical system, and that they will serve to furnish the equations describing any space-time field. Hamilton's principle then becomes, in effect, a convenient abbreviation for the field which on expansion provides the field equations.

Since the Lagrangian density for a field is not associated with a definite mechanical system, it will not necessarily be given as the difference of a kinetic and potential energy density. Instead, we may use any expression for $\mathfrak{L}$ which leads to the desired field equations. Thus, the Lagrangian density for the sound field was obtained previously in Section 11–3 by considering the mechanical system in which the sound was propagated. We were then naturally led to describe the field by using the components of the vector displacement $\boldsymbol{\eta}$ as the generalized coordinates. It will be noted, however, that the field is really a scalar one, since the properties of sound can be discussed completely in terms of the single scalar σ, the fractional change in the density. The quantity σ is therefore the natural generalized coordinate to use if we set out to describe the field by a Lagrangian without reference to any mechanical system. $\mathfrak{L}$ must then be such that Lagrange's equation of motion is the wave equation in σ, Eq. (11–43). It is easily seen that a Lagrangian density in terms of σ which fulfills this requirement is

$$\mathfrak{L} = \frac{1}{2}\left(\frac{\mu_0}{\gamma P_0}\dot{\sigma}^2 - (\nabla\sigma)^2\right) = \frac{\mu_0}{2\gamma P_0}\dot{\sigma}^2 - \frac{1}{2}\sum_k\left(\frac{\partial\sigma}{\partial x_k}\right)^2. \qquad (11\text{–}61)$$

The various derivatives needed for the equation of motion are

$$\frac{\partial\mathfrak{L}}{\partial\sigma} = 0, \qquad \frac{\partial\mathfrak{L}}{\partial\dot{\sigma}} = \frac{\mu_0\dot{\sigma}}{\gamma P_0}, \qquad \frac{\partial\mathfrak{L}}{\partial\left(\frac{\partial\sigma}{\partial x_k}\right)} = -\frac{\partial\sigma}{\partial x_k},$$

so that Lagrange's equation (11–17) becomes

$$\frac{\mu_0}{\gamma P_0}\frac{\partial^2\sigma}{\partial t^2} - \sum_k\frac{\partial^2\sigma}{\partial x_k^2} = 0,$$

which agrees with Eq. (11–43). Note that the Lagrangian density (11–61) is not the same as the density obtained in Section 11–3, Eq. (11–40). Neither of the terms in (11–61) is the mechanical kinetic or potential energy density. But the Lagrangian density does lead to the correct wave equation, and this is all that is required.

Similarly, one could obtain a satisfactory description of the sound field by using a Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(\frac{\gamma P_0}{\mu_0}\pi^2 + (\nabla\sigma)^2\right). \tag{11-62}$$

Eq. (11–62) obviously leads to the proper field equation, but it is not the same as (11–53), nor is it an energy density of the mechanical system.

As a more complicated example of tailoring a Lagrangian density to fit a given field, let us consider the electromagnetic field in the absence of material media. In such case, $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$ in Gaussian units, and Maxwell's equations (1–55) reduce to

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{11-63}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi\mathbf{j}}{c}. \tag{11-64}$$

The first two equations, it will be remembered, serve mainly to define $\mathbf{E}$ and $\mathbf{B}$ in terms of a scalar and vector potential:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{1-58}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1-57}$$

It is the second pair of equations which describe the generation of the fields by external charges and currents, and which may therefore be considered as the desired field equations. The six field components are not very suitable as generalized coordinates, since they are not independent, but can be expressed in terms of the four potential components. We shall therefore use $\mathbf{A}$ and ϕ as the generalized coordinates of the field.

It is now stated that a Lagrangian density which leads to the field equations (11–64) is

$$\mathfrak{L} = \frac{E^2 - B^2}{8\pi} - \rho\phi + \frac{\mathbf{j} \cdot \mathbf{A}}{c}, \tag{11-65}$$

where $\mathbf{E}$ and $\mathbf{B}$ are to be expressed in terms of the potentials by means of Eqs. (1–57, 58). To verify this statement, let us first find the equation of motion corresponding to the coordinate ϕ. The scalar potential itself appears only in the term $-\rho\phi$, and the spatial derivatives of ϕ occur only in E^2. (Note that there are no time derivatives of ϕ anywhere.) Hence we have

$$\frac{\partial \mathfrak{L}}{\partial \phi} = -\rho$$

and

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x_k}\right)} = \frac{E_k}{4\pi} \frac{\partial E_k}{\partial\left(\frac{\partial \phi}{\partial x_k}\right)} = -\frac{E_k}{4\pi},$$

the last step following from Eq. (1–58). Lagrange's equation for ϕ is therefore

$$\frac{1}{4\pi} \sum_k \frac{\partial E_k}{\partial x_k} - \rho = 0$$

or

$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$

which is the first of Eqs. (11–64).

The evaluation of the equations of motion corresponding to the components of **A** is somewhat lengthier. Consider a typical component of **A**, say A_1. The Lagrangian density contains both spatial and time derivatives of A_1, with the component itself occurring in the term $(\mathbf{j} \cdot \mathbf{A})/c$. The required derivatives of $\mathcal{L}$ are therefore

$$\frac{\partial \mathcal{L}}{\partial A_1} = \frac{j_1}{c},$$

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_1} = \frac{E_1}{4\pi} \frac{\partial E_1}{\partial \dot{A}_1} = -\frac{E_1}{4\pi c},$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_1}{\partial x_2}\right)} = -\frac{1}{4\pi} B_3 \frac{\partial B_3}{\partial\left(\frac{\partial A_1}{\partial x_2}\right)} = \frac{B_3}{4\pi},$$

and finally,

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_1}{\partial x_3}\right)} = -\frac{B_2}{4\pi}.$$

Combining these expressions, Lagrange's equation for A_1 appears as

$$\frac{1}{4\pi}\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}\right) - \frac{1}{4\pi c}\frac{dE_1}{dt} - \frac{j_1}{c} = 0. \tag{11–66}$$

Equation (11–66) is readily recognized as the 1-component of the remaining Maxwell equation

$$(\nabla \times \mathbf{B}) - \frac{1}{c}\frac{d\mathbf{E}}{dt} = \frac{4\pi \mathbf{j}}{c},$$

the other components of which are the equations of motion for A_2 and A_3. Thus the four field equations obtained from the Lagrangian density (11–65) are identical with the Maxwell's equations (11–64).*

The charge and current density are connected, of course, by the relation

$$\mathbf{j} = \rho \mathbf{v}, \tag{11–67}$$

where $\mathbf{v}$ is the velocity of the charges, a function of position in space. The volume integral of the Lagrangian density (11–65) is the total Lagrangian for the electromagnetic field, and by virtue of Eq. (11–67) can be written

$$L = \int \left\{ \frac{E^2 - B^2}{8\pi} - \rho \left(\phi - \frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) \right\} dV. \tag{11–68}$$

It will be noted that the combination of ϕ and $\mathbf{A}$ in the parentheses appears also in the Lagrangian Eq. (1–61) for a charged particle in an electromagnetic field. Indeed, we can show that this part of the field Lagrangian corresponds exactly to the generalized potential part of the particle Lagrangian.

The term "particle" indicates that the mass and charge are concentrated at a point in space. Hence the charge density corresponding to a particle must be zero everywhere except at the position of the particle. There it must be infinite, but in such a manner that the volume integral of the charge density is equal to the total charge on the particle. The mathematical function satisfying these peculiar requirements is a multiple of the well-known volume δ-function introduced by Dirac, and defined by the conditions:

$$\begin{aligned} \delta(\mathbf{r} - \mathbf{r}_1) &= 0, \qquad \mathbf{r} \neq \mathbf{r}_1 \\ \int \delta(\mathbf{r} - \mathbf{r}_1)\, dV &= 1, \end{aligned} \tag{11–69}$$

where $\mathbf{r}_1$ is the position vector of the particle. An obvious property of the δ-function is that if $f(\mathbf{r})$ is any function of position, then

$$\int f(\mathbf{r})\, \delta(\mathbf{r} - \mathbf{r}_1)\, dV = f(\mathbf{r}_1). \tag{11–70}$$

* In some respects the electromagnetic field is an unfortunate example, for it presents a number of unique difficulties. It was mentioned that $\mathfrak{L}$ does not contain $\dot{\phi}$. Hence there is no canonical momentum conjugate to ϕ, and the Hamiltonian formulation of Section 11–4 falls through. Essentially, the source of the difficulty is that the scalar and vector potentials are not entirely independent quantities, but are connected by the so-called *gauge condition*. The gauge requirement acts as a supplementary condition which can be used, in effect, to eliminate one of the generalized coordinates, so that the field is described only by independent coordinates. For further details, see G. Wentzel, *Introduction to the Quantum Theory of Fields*.

The density function for a group of n particles can therefore be represented as

$$\rho(\mathbf{r}) = \sum_{i=1}^{n} q_i \, \delta(\mathbf{r} - \mathbf{r}_i), \tag{11-71}$$

where q_i is the charge of the ith particle and $\mathbf{r}_i$ its position vector.

Let us now evaluate the integral

$$\int \rho(\mathbf{r}) \left\{ \phi(\mathbf{r}) - \frac{\mathbf{v} \cdot \mathbf{A}(\mathbf{r})}{c} \right\} dV$$

for the charge density (11-71). By Eq. (11-70) the integral is then equal to

$$\sum_i q_i \left(\phi(\mathbf{r}_i) - \frac{\mathbf{v}_i \cdot \mathbf{A}(\mathbf{r}_i)}{c} \right),$$

so that when the charges and currents arise from a system of particles the total Lagrangian for the field becomes

$$L = \int \frac{E^2 - B^2}{8\pi} \, dV - \sum_i q_i \left(\phi_i - \frac{\mathbf{v}_i \cdot \mathbf{A}_i}{c} \right). \tag{11-72}$$

Here the subscripts indicate that ϕ and $\mathbf{A}$ are to be evaluated at the positions of the particles. Comparison with Eq. (1-61) now shows that the summation in (11-72) is *exactly* that part of the Lagrangian for the n particles which produces the electromagnetic forces acting on the particles! We can, in fact, combine the two Lagrangians by adding the kinetic energy of the particles to (11-72):

$$L = \int \frac{E^2 - B^2}{8\pi} \, dV - \sum_i q_i \left(\phi_i - \frac{\mathbf{v}_i \cdot \mathbf{A}_i}{c} \right) + \frac{1}{2} \sum_i m_i v_i^2. \tag{11-73}$$

An alternative form for Eq. (11-73) is

$$L = \int \left\{ \frac{E^2 - B^2}{8\pi} - \sum_i q_i \, \delta(\mathbf{r} - \mathbf{r}_i) \left(\phi - \frac{\mathbf{v}_i \cdot \mathbf{A}}{c} \right) \right\} dV + \frac{1}{2} \sum_i m_i v_i^2. \tag{11-73'}$$

In Eq. (11-73) or (11-73′) we have a total Lagrangian which describes both the electromagnetic field on the one hand, and the mechanical motion of the n particles on the other. It is a function of the generalized coordinates ϕ, $\mathbf{A}$ with continuous space indices x_1, x_2, x_3, *and* the particle generalized coordinates $\mathbf{r}_i$ distinguished by a discrete index i. A single Hamilton's principle thus suffices for both systems! Variation with respect to the potentials produces the Maxwell equations of the field, while variation with respect to the particle coordinates results in the particle

equations of motion. Note that the first term in Eq. (11–73) represents the Lagrangian for the field in the absence of the charged particles. Similarly, the last term represents the Lagrangian for the particles in the absence of the field. The middle term then furnishes the mutual interaction between particle and field.

Expressing fields in terms of a variational principle formulation thus results in a description that is compact and elegant. One may well ask, however, what practical advantage the formulation has over the direct use of the field equations. The most important applications actually lie outside the domain of classical physics, but may be mentioned here briefly.

First, the Lagrangian formulation provides a convenient technique for inventing new types of fields and investigating their properties. The possible terms in the Lagrangian density are closely limited by the requirements that $\mathfrak{L}$ contain only the coordinates and their first spatial and time derivatives, and that $\mathfrak{L}$ be a Lorentz invariant. For example, if there is only one type of generalized coordinate η, which must be a world scalar (or pseudoscalar), there are only three possible terms meeting these requirements:

$$\eta, \qquad \sum_\mu \left(\frac{\partial \eta}{\partial x_\mu}\right)^2, \qquad \sum_\mu A_\mu \frac{\partial \eta}{\partial x_\mu},$$

where A_μ is an external world vector (or pseudovector). Any Lagrangian density for a scalar field must therefore be composed of combinations of these terms. One can thus examine many of the general properties of such a scalar field without knowing the physical mechanism producing the field. This is the technique currently used in much of the theoretical work on meson fields.

The second application has to do with the quantization of fields. It has already been remarked that the transition from classical to quantum theory can be stated only in terms of the canonical variables describing a system. Thus it was pointed out that the classical Poisson brackets of the canonical coordinates, or functions of them, correspond to the quantum commutation relations. In effect, we know how to quantize a system only when we can speak of it in mechanical terms. If it is desired to construct a quantum theory of the electromagnetic field, or of any other field, we must first obtain a description of the field in the language of mechanics. The Lagrangian and Hamiltonian formulations presented in this chapter form the basis for such a description.

SUGGESTED REFERENCES

J. C. SLATER AND N. H. FRANK, *Mechanics.* In discussing the mechanics of continuous systems, we have stopped at the equations of motion, and have not considered their solutions in detail. To treat the vibrations of strings, membranes, fluids, or solids would require a separate volume in itself. The text by Slater and Frank devotes almost half the space to such questions, and forms a readable, if somewhat elementary, introduction to the subject. In particular, the transition from the discrete chain to the continuous string is discussed in Chapter VII for *transverse* vibrations.

LORD RAYLEIGH, *The Theory of Sound.* This treatise naturally contains much material on the vibrations of continuous bodies. A discussion of the wave equation for the propagation of sound in gases will be found in Chapter XI, Volume 2, where the question of adiabatic vs. isothermal motion of the gas is examined in great detail.

G. WENTZEL, *Introduction to the Quantum Theory of Fields.* No single reference can be given which contains a detailed and comprehensive treatment of the classical mechanics of fields. Because the classical theory was usually developed as a preliminary to quantization of the field, most discussions are to be found in works on quantum mechanics. The best source is probably the excellent and well written volume of Wentzel, especially Chapter 1. Valuable material is also given in Chapters XIII and XIV of *Quantum Mechanics* by L. I. Schiff. The latter chapter, in particular, is devoted to the electromagnetic field. An earlier reference is Section 9 of the Appendix to W. Heisenberg's *The Physical Principles of the Quantum Theory.* The pioneering work in the theory of fields, and still quite valuable, is the paper by W. Heisenberg and W. H. Pauli, Zeitschr. f. Phys. **56,** 1 (1929).

EXERCISES

1. (a) The transverse vibrations of a stretched string can be approximated by a discrete system consisting of equally spaced mass points located on a weightless string. Show that if the spacing is allowed to go to zero, the Lagrangian approaches the limit

$$L = \frac{1}{2}\int \left[\mu\dot{\eta}^2 - T\left(\frac{\partial\eta}{\partial x}\right)^2\right] dx$$

for the continuous string, where T is the fixed tension. What is the equation of motion if the density μ is a function of position?

(b) Obtain the Lagrangian for the continuous string by finding the kinetic and potential energies corresponding to transverse motion. The potential energy can be obtained from the work done by the tension force in stretching the string in the course of the transverse vibration.

2. Obtain Hamilton's equations of motion for a continuous system from the modified Hamilton's principle (11–58), following the procedure of Section 7–4.

3. Show that if ψ and ψ^* are taken as two independent field variables, the Lagrangian density

$$\mathfrak{L} = \frac{h^2}{8\pi^2 m} \nabla\psi \cdot \nabla\psi^* + V\psi^*\psi + \frac{h}{2\pi i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*),$$

leads to the Schrödinger equation

$$-\frac{h^2}{8\pi^2 m}\nabla^2\psi + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t},$$

and its complex conjugate. What are the canonical momenta? Obtain the Hamiltonian density corresponding to $\mathfrak{L}$.

4. Show that

$$G_i = -\int \sum_k \pi_k \frac{\partial \eta_k}{\partial x_i} dV$$

is a constant of the motion if the Hamiltonian density is not an explicit function of position. The quantity G_i can be identified as the total linear momentum of the field along the x_i direction. The similarity of this theorem with the usual conservation theorem for linear momentum of discrete systems (cf. Section 8–6) is obvious.

5. A Lagrangian density for the electromagnetic field is given by the relativistic covariant form:

$$\mathfrak{L} = -\frac{1}{16\pi}\sum_{\mu,\nu}\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right)^2 - \frac{1}{8\pi}\sum_\mu\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2 + \sum_\mu \frac{j_\mu A_\mu}{c},$$

where A_μ is the potential four-vector and j_μ a four-vector with components $\mathbf{j}$ and $i\rho c$. Show that the Lagrangian density leads directly to the wave equations for the vector potential:

$$\Box^2 A_\mu = \frac{4\pi j_\mu}{c}.$$

Show also that this Lagrangian density is identical with that given in the text, except for the middle term (which has zero value anyhow as a result of the gauge condition).